Indian Statistical Institute

Final Examination 2014-2015

B.Math Third Year

Complex Analysis

Time : 3 Hours Date : 05.11.2014 Maximum Marks : 100 Instructor : Jaydeb Sarkar

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(i) Answer all questions. (ii) $B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$. (iii) $P_r(z_0) := \{z \in \mathbb{C} : 0 < |z - z_0| < r\}$. (iv) $C_r(z_0) := \{z \in \mathbb{C} : |z - z_0| = r\}$.

Q1. (15 marks) Prove or disprove that:

(i) $f(z) = |z|$ has a primitive in $\mathbb{C}$.

(ii) there exists $f \in \text{Hol}(B_1(0))$ such that $f(\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$ and $f(z) \neq 0$ for some z.

(iii) there exists $f \in \text{Hol}(B_1(0))$ such that $|f(z)| = \exp(|z|)$ for all $z \in B_1(0)$.

Q2. (15 marks) Let $f \in \text{Hol}(B_1(0))$ and $f(0) = f'(0) = 0$, and $|f(z)| \leq 1$ for all $z \in B_1(0)$. Prove that $|f''(0)| \leq 2$. When can equality occur?

Q3. (15 marks) Let $f \in \text{Hol}(B_e(0))$ and $|f(z)| < 1$ for all $z \in B_e(0)$. Determine the cardinality of $\{z \in B_e(0) : f(z) = z\}$.

Q4. (15 marks) Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of complex numbers and $f \in \text{Hol}(\mathbb{C} \setminus \{\alpha_n\}_{n=1}^{\infty})$. Prove that f is a constant multiple of r, where r is a rational function and $|f(z)| \leq |r(z)|$ for all $z \in \mathbb{C} \setminus \{\alpha_n\}_{n=1}^{\infty}$.

Q5. (15 marks) Prove that the upper half plane and $B_1(0)$ are biholomorphically equivalent.

Q6. (15 marks) Use the residue theorem to compute the following integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}.$$

Q7. (15 marks) Let $f \in \text{Hol}(P_1(z_0))$ and Real $f \geq 0$ on $P_1(z_0)$. Prove that z_0 is a removable singularity of f.